Lattice Practices
Solvers II – Preconditioning
October 15, 2015 – FZ Jülich

The scope of this exercise is to explore and play around with some options of preconditioning. Again demos for each task can be found in the matlab folder for this exercise. The questions given on this sheet are meant to be discussed with your fellow lattice practitioners while inspecting the demo.

Task 1  Preconditioned Conjugate Gradients
We consider the system $Ax = b$ with $A$ the discrete Laplacian (see yesterday’s exercises).

1. The first and most simple preconditioning one can use is diagonal scaling (a.k.a. Jacobi) preconditioning. In here

$$ S = D^{-1}, \quad D = \text{diag}(A). $$

- Why does this preconditioning idea fail miserably (It does not help at all!)? (Hint: Inspect the diagonal of $A$.)
- Modify the matrix $A$ by a random scaling:

$$ A = D_z A D_z, \quad D_z = \text{diag}(z_1, \ldots, z_n), $$

$z_i$ uniformly distributed in $(0, 1)$.

Compare the results with the ones obtained for the original matrix.

2. Preconditioning by SSOR (symmetric successive over-relaxation)

$$ x^{(k+1/2)} = x^{(k)} + \left( \frac{1}{2} D + L \right)^{-1} r^{(k)} $$

$$ x^{(k+1)} = x^{(k+1/2)} + \left( \frac{1}{2} D + U \right)^{-1} r^{(k)} $$

reduces the condition number significantly. You can modify the over-relaxation parameter $\omega \in (0, 2)$ and look at the impact on preconditioning efficiency.
3. Compare the spectrum of the SSOR preconditioned matrix with the one you obtained in task 1 of yesterday’s exercise.

4. A class of black-box preconditioners available are the thresholded ILU methods.
   - Take a look at the fill-in created by ILU, comparing the number of non-zeros in $A$ and $\tilde{L}, \tilde{U}$. Play around with the threshold and see what happens.

5. **Bonus**: Consider the situation, where the spectrum of $A$ (hermitian positive definite) has the following structure. All the eigenvalues but one of $A$ are contained in an interval $[a, b]$, the remaining eigenvalue is located at $c \gg b$ (or $0 < c \ll a$). Hence the condition number $\kappa$ is given by
   \[
   \kappa = \frac{c}{a} \quad \text{or} \quad \kappa = \frac{b}{c}.
   \]
   Why do you expect the CG method to converge much faster than predicted by the convergence theory? (Hint: Think about the interpretation of CG as approximating $A^{-1}$ on the spectrum of $A$ by a polynomial!)
   - Can you come up with a simple linear system $Ax = b$ to test the situation? (Hint: Prescribe the eigenvalues!)
   - Especially when using diverging preconditioners situations like the one described can occur, why? Assume that the preconditioner only diverges on a small subspace of eigenmodes.

**Task 2  Preconditioned GMRES**

In order to show properties of the GMRES iteration we consider an example from Lattice QCD. The system matrix $A$ is given by the Wilson discretization of the Dirac equation on a $4^4$ lattice at $\beta = 6$ with an additive mass shift. The system matrix is non-hermitian with its eigenvalues in the right half-plane.

1. The first preconditioner to try for this problem is a domain decomposition approach with $2^4$ blocks (including all 12 variables on each lattice site).

2. Again we can apply the ILU preconditioner with a set threshold. Change the threshold and observe how the fill-in and the performance of the preconditioner changes.

3. The last preconditioner to be used is the odd-even preconditioner. In here we solve $A_{ee}x_e = b_e$ by BiCGstab with a fixed accuracy. The method does not converge for an accuracy of $10^{-1}$. What happened? What is the cure?
Task 3  Multigrid
For the discrete Laplacian, the symbolic stencil notation above describes a restriction operator $R$ and a prolongation operator $P$.

1. Explore multigrid:
   - Run the multigrid method for $N = 7, 15, 31, 63$ and $127$.
   - How does the number of iterations scale with $N$?
   - Compare with CG.
   - Determine the smoothing iteration used by inspecting the code.

2. Use multigrid as a preconditioner to GMRES:
   - How much do you gain as compared to “stand-alone” multigrid?
   - Why do we use GMRES and not CG, here?

3. Bonus* Explain why the multigrid idea is more difficult to apply to the gauge Laplacian (and to the Wilson-Dirac system).

Task 4  Preconditioned BiCGstab
To conclude we demonstrate the typical convergence behavior of preconditioned BiCGstab, applied again to the $4^4$ Lattice Dirac Wilson operator. We use the same preconditioner as for the GMRES method. Again, play around with the thresholds and settings and observe the efficiency of the preconditioners.